JOURNAL OF APPROXIMATION THEORY **81**, 136–140 (1995)

# Note

## A Note on Two Moduli of Smoothness

GANCHO T. TACHEV

*Department of Mathematics, University of Architecture, Civil Engineering and Geodesy, 1421 Sofia, Bulgaria*

*Communicated by Zeev Ditzian*

Received November 2, 1992; accepted in revised form December 19, 1993

We prove that for $f \in L_p[-1, 1]$, $0<p<1$ the modulus of smoothness $\tau_k(f, \Delta_n)_{p,p}$ introduced by Ivanov and Ditzian–Totik modulus of smoothness $\omega_\varphi^k(f, n^{-1})_p$ are equivalent. © 1995 Academic Press, Inc.

### 1. INTRODUCTION

The modulus of smoothness $\tau_k(f, \Delta_n)_{p,p}$ is defined by

$$\tau_k(f, \Delta_n)_{p,p} = \|\omega_k(f, \cdot, \Delta_n(\cdot))_p\|_{L_p[-1,1]}, \tag{1.1}$$

where the local $L_p$ modulus of continuity is defined by

$$\omega_k(f, x, \Delta_n(x))_p = \left((2\Delta_n(x))^{-1} \int_{-\Delta_n(x)}^{\Delta_n(x)} |\delta_h^k f(x)|^p \, dh\right)^{1/p}.$$

Here $k$, $n \in N$—the set of natural numbers,

$$\Delta_n(x) = n^{-1}(1-x^2)^{1/2} + n^{-2}, \qquad f \in L_p[-1, 1], 0<p \leqslant \infty$$

and the finite difference $\delta_h^k f(x)$ is defined as

$$\sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} f(x+rh) \qquad \text{if} \quad x, x+kh \in [-1, 1] \text{ and as 0, otherwise.}$$



0021-9045/95 $6.00

This modulus was introduced by Ivanov in [5]. If $\varphi(x)=(1-x^2)^{1/2}$ the Ditzian–Totik modulus of smoothness of $f\in L_p[-1,1]$ is defined by

$$\omega_\varphi^k(f,n^{-1})_p=\sup_{0<h\leqslant n^{-1}}\|\Delta_{h\varphi(\cdot)}^k f(\cdot)\|_{L_p[-1,1]}$$

$$\Delta_{h\varphi(x)}^k=\begin{cases}\sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r}f(x-kh\varphi(x)/2+rh\varphi(x)), & \\ \qquad x\pm kh\varphi(x)/2\in[-1,1] & \\ 0, \qquad \text{otherwise.} & \end{cases} \tag{1.2}$$

For $1\leqslant p<\infty$ the equivalence between moduli (1.1) and (1.2) was proved via the $K$-functional (see [4,6]). However, $\omega_\varphi^k(f,n^{-1})_p$ cannot be equivalent to the appropriate $K$-functional when $0<p<1$. Our main result is the following

THEOREM 1. *Let $k\in N$, $0<p<1$. Then for every $f\in L_p[-1,1]$ and $n\geqslant M$ (with $M$ a constant depending only on $p$ and $k$) there are positive constants $c_1$ and $c_2$, dependent only on $p$ and $k$, such that*

$$c_1\omega_\varphi^k(f,n^{-1})_p\leqslant\tau_k(f,\Delta_n)_{p,p}\leqslant c_2\omega_\varphi^k(f,n^{-1})_p. \tag{1.3}$$

COROLLARY. *For $0<\alpha<k$, $f\in L_p[-1,1]$, $0<p\leqslant\infty$ the following are equivalent*

(1) $E_n(f)_p=(n^{-\alpha})$

(2) $\tau_k(f,\Delta_n)_{p,p}=O(n^{-\alpha})$

(3) $\omega_\varphi^k(f,n^{-1})_p=O(n^{-\alpha})$,

*where $E_n(f)_p$ denotes the best $L_p$ approximation of $f$ by algebraic polynomials of $n$th degree.*

The case $1\leqslant p\leqslant\infty$ was considered by Ivanov and Ditzian and Totik (see [4–6]). The case $0<p<1$ follows from Theorem 1, [7], [8]. Recently, Ditzian *et al.* [3] has proved the equivalence $(1)\Leftrightarrow(3)$ when $0<p<1$. This note gives the answer of their question about the relation between $\tau_k(f,\Delta_n)_{p,p}$ and $\omega_\varphi^k(f,n^{-1})_p$ for $0<p<1$.

## 2. PROOF OF THEOREM 1

Let $n$ be sufficiently large and fixed.

To prove Theorem 1 we shall use two direct estimations for approximation of function $f\in L_p[-1,1]$, $0<p<1$ by piecewise polynomial functions $S_{k-1,n}(x)$ and $L_n(f,x)$. The function $S_{k-1,n}(x)$ is defined by

$$S_{k-1,n}(x)=P_i(x), \qquad \text{for } x\in[x_{i-1},x_i),\ i=1,2,...,n_0, \tag{2.1}$$

where $P_i(x)$ is the algebraic polynomial of degree $\leqslant k-1$ which is the best $L_p$ approximation to $f$ on the interval $[x_{i-1}, x_{i+1})$ and the knots $\{x_i\}$ are chosen to satisfy $x_0=-1$, $x_{i+1}=x_i+\Delta_n(x_i)$, $x_{n_0}=1$, $x_i=-1$ for $i<0$ and $x_i=1$ for $i>n_0$. It is clear that $n_0=O(n^2)$.

The following estimation for approximation by $S_{k-1,n}$ was proved in [7]

$$\|f-S_{k-1,n}\|_{L_p[-1,1]} \leqslant c(k,p)\,\tau_k(f, 180\,\Delta_n)_{p,p}, \tag{2.2}$$

where $c(k,p)$ is a positive constant which depends only on $p$ and $k$ and may differ at each occurence. We can take the constant 180 out of the modulus as its was shown in (2.21) in [9].

The function $L_n(f,x)$ is defined by

$$L_n(f,x)=q_i(x) \qquad \text{for} \quad x\in[\xi_{i-1}, \xi_i],\ i=1, ..., n, \tag{2.3}$$

where $q_i(x)$ is the algebraic polynomial of degree $\leqslant k-1$, which is a near-best $L_p$ approximation to $f$ on the interval $[\xi_{i-4}, \xi_{i+3}]\cap[-1,1]$, $i=1, ..., n-1$, and the knots $\{\xi_i\}$ are defined in [1].

The following estimation for approximation by $L_n(f)$ was proved in [1]

$$\|f-L_n(f)\|_{L_p[-1,1]} \leqslant c(k,p)\,\omega_\varphi^k(f, n^{-1})_p, \qquad n\geqslant 10. \tag{2.4}$$

It is easy to see that for each $f\in L_p[-1,1]$, $0<p<1$

$$\omega_\varphi^k(f, n^{-1})_p \leqslant c(k,p)\,\|f\|_p \qquad \text{and} \qquad \tau_k(f, \Delta_n)_{p,p} \leqslant c(k,p)\,\|f\|_p \tag{2.5}$$

for sufficiently large $n\in N$ (see [3, 8]). Now we are ready to prove Theorem 1. We start with the proof of the first inequality in (1.3). Let $\beta=\beta(k)$ and $0<1/n<\beta<1$. Under this assumption we can choose $\beta$ sufficiently small, such that

$$[x-kh\varphi(x)/2, x+kh\varphi(x)/2]\subseteq[x_{i-2}, x_{i+1}] \qquad \text{for} \quad x\in[x_{i-1}, x_i].$$

The last imbedding follows from the fact that adjacent intervals $[x_{i-1}, x_i]$ have comparable lengths (see [6]). Using the inequality $(\sum x_i)^p \leqslant \sum x_i^p$ and the identity $\Delta^k_{h\varphi(x)} S_{k-1,n}(x)=\Delta^k_{h\varphi(x)}(S_{k-1,n}-P_i)(x)$ we have

$$\begin{aligned}
&\omega_\varphi^k(f, \beta n^{-1})_p^p \\
&\quad = \sup_{0<h\leqslant \beta n^{-1}} \sum_{i=1}^{n_0} \int_{x_{i-1}}^{x_i} |\Delta^k_{h\varphi(x)}(f-P_i)(x)|^p\,dx \\
&\quad \leqslant c(k,p) \sup_{0<h\leqslant \beta n^{-1}} \sum_{i=1}^{n_0} \sum_{r=0}^{k} \int_D |(f-P_i)(x-kh\varphi(x)/2+rh\varphi(x))|^p\,dx,
\end{aligned} \tag{2.6}$$

where $D=[x_{i-1}, x_i]\cap\{x \mid x\pm kh\varphi(x)/2\in[-1,1]\}$.

According to (2.1) after simple change of variables $x+jh\varphi(x)/2=y$ for $j\in[-k,k]$ in the integral in (2.6) it is easy to verify that $dx=|J|\,dy$ where $|J|\leqslant 2$. From (2.5) and (2.6) we obtain

$$\omega_\varphi^k(f,\beta n^{-1})_p^p\leqslant c(k,p)\sum_{i=1}^{n_0}\|f-P_i\|_{L_p[x_{i-2},x_{i+1}]}^p$$
$$\leqslant c(k,p)\,\|f-S_{k-1,n}\|_p^p\leqslant c(k,p)\,\tau_k(f,\Delta_n)_{p,p}^p \tag{2.7}$$

In the last two inequalities we used (2.1), (2.2), the fact that best $L_p$ approximation to $f$ on the intervals $[x_{i-1},x_i]$, $[x_{i-2},x_{i-1}]$ and $[x_i,x_{i+1}]$ is a near-best $L_p$ approximation to $f$ on the interval $[x_{i-2},x_{i+1}]$ with a constant, dependent only on $p$ and $k$ (see [2]) and the method of proving of (2.2) (see [7]).

In order to complete the proof of the left inequality in (1.3) we have

$$\omega_\varphi^k(f,n^{-1})_p^p=\omega_\varphi^k(f,\beta(n\beta)^{-1})_p^p\leqslant\omega_\varphi^k(f,\beta[n\beta]^{-1})_p^p$$
$$\leqslant c(k,p)\,\tau_k(f,\Delta_{[n\beta]})_{p,p}^p\leqslant c(k,p)\,\tau_k(f,\Delta_n)_{p,p}^p$$

The proof of second inequality in (1.3) is similar. Let $\gamma=\gamma(k)$ and $0<\gamma<1$. From the fact that we can take the constant out of the modulus $\tau_k(f,\Delta_n)_{p,p}$ ([9]) we get

$$\tau_k(f,\Delta_n)_{p,p}^p\leqslant c(k,p)\,\tau_k(f,\gamma\Delta_n)_{p,p}^p$$
$$\leqslant c(k,p)\sum_{i=1}^{n}\int_{\xi_{i-1}}^{\xi_i}\frac{1}{2\gamma\,\Delta_n(x)}\int_{-\gamma\Delta_n(x)}^{\gamma\Delta_n(x)}|\delta_h^k(f-q_i)(x)|^p\,dh\,dx$$
$$\leqslant c(k,p)\sum_{i=1}^{n}\sum_{j=0}^{k}\int_{\xi_{i-1}}^{\xi_i}\frac{1}{2\gamma\,\Delta_n(x)}\int_{E(i,x)}|(f-q_i)(x+jh)|^p\,dh\,dx, \tag{2.8}$$

where $E(i,x)=[-\gamma\,\Delta_n(x),\gamma\,\Delta_n(x)]\cap\{h\mid x+kh\in[-1,1]\}$, for fixed $x\in[\xi_{i-1},\xi_i]$. We chose $\gamma$ sufficiently small such that

$$x+kh\in[\xi_{i-2},\xi_{i+1}],\qquad\text{for}\quad x\in[\xi_{i-1},\xi_i]\qquad\text{and}\qquad h\in E(i,x).$$

Let us define

$$\lambda_i=\max\{(2\gamma\Delta_n(\xi_{i-1}))^{-1},(2\gamma\Delta_n(\xi_i))^{-1}\}\qquad\text{for}\quad 1\leqslant i\leqslant n.$$

From the definitions of the points $\{\xi_i\}$ (see Lemma 3.1 in [1]) and $\Delta_n(x)$ it follows that $(\xi_i-\xi_{i-1})\,\lambda_i\leqslant c$, where $c$ is a constant, independent of $n$. Using this observation and (2.8) we get

$$\tau_k(f,\Delta_n)_{p,p}^p\leqslant c(k,p)\sum_{i=1}^{n}\sum_{j=0}^{k}\lambda_i\int_{\xi_{i-1}}^{\xi_i}\int_{E(i,x)}|(f-q_i)(x+jh)|^p\,dh\,dx \tag{2.9}$$

After simple change of variables $x+jh=y$, $h=v$ and changing the order of integration in (2.9) we obtain

$$\tau_k(f, \Delta_n)_{p,p}^p \leqslant c(k,p) \sum_{i=1}^{n} (\xi_i - \xi_{i-1}) \lambda_i \|f-q_i\|_{L_p[\xi_{i-2}, \xi_{i+1}]}^p$$
$$\leqslant c(k,p) \sum_{i=1}^{n} \|f-q_i\|_{L_p[\xi_{i-2}, \xi_{i+1}]}^p. \tag{2.10}$$

Now the proof of (1.3) follows from (2.10), (2.4) and (4.10) in [1]. This completes the proof of Theorem 1.


## ACKNOWLEDGMENTS

The author expresses his gratitute to Professor K. G. Ivanov and to Professor Z. Ditzian for the valuable suggestion and helpful remarks.


## REFERENCES


1. R. A. DEVORE, D. LEVIATAN, AND X. M. YU Polynomial approximation in $L_p$ $(0<p<1)$, *Constr. Approx.* **8** (1992), 187–201.
2. R. A. DEVORE AND V. POPOV, Interpolation of Besov spaces, *Trans. Amer. Math. Soc.* **305** (1988), 397–414.
3. Z. DITZIAN, D. JIANG, AND D. LEVIATAN, Inverse theorem for best polynomial approximation in $L_p$, $0<p<1$, *Proc. Amer. Math. Soc.*, to appear.
4. Z. DITZIAN AND V. TOTIK, "Moduli of Smoothness," Springer Series in Computational Mathematics, Vol. 9, Springer-Verlag, New York/Berlin.
5. K. G. IVANOV, Direct and converse theorem for the best algebraic approximation in $C[-1,1]$ and $L_p[-1,1]$, *C. R. Acad. Bulgare Sci.* **33** (1980), 1309–1312.
6. K. G. IVANOV, A characterization of weighted Peetre $K$-functionals, *J. Approx. Theory* **56** (1989), 185–211.
7. G. T. TACHEV, A direct theorem for the best algebraic approximation in $L_p[-1,1]$, $(0<p<1)$, *Math. Balk.* **4** (1990), 381–390.
8. G. T. TACHEV, A converse theorem for the best algebraic approximation in $L_p[-1,1]$, $(0<p<1)$, *Serdica* **17** (1991), 161–166.
9. G. T. TACHEV, Approximation by Kantorovich-Bernstein polynomials in $L_p$ $(0<p<1)$-metric, *Approx. Theory Appl.*, to appear.